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Janossy Densities. I. Determinantal Ensembles

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We derive an elementary formula for Janossy densities for determinantal point processes with a finite rank projection-type kernel. In particular, for $\beta=2$ polynomial ensembles of random matrices we show that the Janossy densities on an interval $I \subset \mathbb{R}$ can be expressed in terms of the Christoffel–Darboux kernel for the orthogonal polynomials on the complement of I.

KEY WORDS: Random matrices; orthogonal polynomials; Janossy densities; Riemann–Hilbert problem.

1. INTRODUCTION

We consider an ensemble of n particles on a measure space (X, μ) with the joint distribution density (with respect to the product measure $\mu^{\otimes n}$) given by the formula

$$p(x_1,\dots, x_n) = \mathrm{const}_n \cdot \det(\phi_j(x_k))_{j,\,k=1,\dots,n} \det(\psi_j(x_k))_{j,\,k=1,\dots,n}. \tag{1}$$

Here $\phi_k(x), \psi_k(x)$, $k=1,\dots, n$, are some functions on X and const_n is the normalization constant

$$\begin{aligned}\mathrm{const}_n^{-1} &= \int_{X^n} \det(\phi_j(x_k))_{j,\,k=1,\dots,n} \det(\psi_j(x_k))_{j,\,k=1,\dots,n} \prod_{j=1,\dots,n} \mu(dx_j)\\ &= n! \det\left(\int_X \phi_i(x)\,\psi_j(x)\,\mu(dx)\right)_{i,\,j=1,\dots,n}\end{aligned} \tag{2}$$

where $X^n = X \times \cdots \times X$ (n times). Ensembles of this form were introduced in refs. 1 and 30. In the special cases when $X=\mathbb{R}$, $\phi_i=\psi_i=x^{i-1}$, and

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$X = \{z \in \mathbb{C} \mid |z| = 1\}$, $\phi_i = \bar{\psi}_i = z^{i-1}$, such ensembles were extensively studied in Random Matrix Theory much earlier under the general name *unitary ensembles*, see ref. 21 for details. An example of the form (1) which is different from random matrix ensembles was considered in ref. 22.

Let us assume that we can biorthogonalize $\{\phi_j\}_{j=1,\dots,n}$ and $\{\psi_j\}_{j=1,\dots,n}$ with respect to the pairing

$$\langle \phi, \psi \rangle = \int_X \phi(x)\, \psi(x)\, \mu(dx).$$

In other words, suppose that we can find functions $\xi_k(x), \eta_k(x), k = 1,\dots, n$ such that

$$\xi_k \in \operatorname{Span}(\phi_j, j = 1,\dots, n), \quad \eta_k \in \operatorname{Span}(\psi_j, j = 1,\dots, n), \qquad \langle \xi_k, \eta_m \rangle = \delta_{km}.$$

The families $\{\xi_j\}$ and $\{\eta_j\}$ are called biorthogonal bases in $\operatorname{Span}(\phi_j, j = 1,\dots, n)$ and $\operatorname{Span}(\psi_j, j = 1,\dots, n)$ considered as subspaces in $L^2(X, \mu)$. Then the distribution (1) can be rewritten as[1,30]

$$p_n(x_1,\dots, x_n) = \frac{1}{n!} \det(K(x_i, x_j))_{i, j = 1,\dots, n}, \tag{3}$$

with

$$K(x, y) = \sum_{j=1}^{n} \xi_j(x)\, \eta_j(y). \tag{4}$$

One of the particularly nice properties of the ensemble (1), (3) is that one can explicitly calculate the correlation functions

$$\rho_k(x_1,\dots, x_k) := \frac{n!}{(n-k)!} \int_{X^{n-k}} p(x_1,\dots, x_k, x_{k+1},\dots, x_n)\, \mu(dx_{k+1}) \cdots \mu(dx_n)$$

which still have a determinantal form with the same kernel $K(x, y)$:

$$\rho_k(x_1,\dots, x_k) = \det(K(x_i, x_j))_{i, j = 1,\dots, k}. \tag{5}$$

If μ is supported by a discrete set of points then the probabilistic meaning of the k-point correlation function is that of the probability to find a particle at each of k sites $x_1, x_2,\dots, x_k$. In other words,

$$\rho_k(x_1, x_2,\dots, x_k)\, \mu(x_1) \cdots \mu(x_k)$$
$$= \Pr\{\text{there is a particle at each of the points } x_i,\ i = 1,\dots, k\}.$$

Analogously, if $X \subset \mathbb{R}$ and μ is absolutely continuous with respect to the Lebesgue measure then

$$\rho_k(x_1, x_2, \ldots, x_k)\, \mu(dx_1) \cdots \mu(dx_k)$$
$$= \Pr\{\text{there is a particle in each infinitesimal interval } (x_i, x_i + dx_i)\}.$$

In general, random point processes with the k-point correlation functions of the determinantal form (5) are called determinantal or fermion (see e.g., ref. 27).

So-called Janossy densities $\mathcal{J}_{k,I}(x_1, \ldots, x_k)$, $k = 0, 1, 2, \ldots$, describe the distribution of the particles in a subset I of X. If $X \subset \mathbb{R}$ and μ is absolutely continuous with respect to the Lebesgue measure then

$$\mathcal{J}_{k,I}(x_1, \ldots, x_k)\, \mu(dx_1) \cdots \mu(dx_k)$$
$$= \Pr\{\text{there are exactly } k \text{ particles in } I, \text{ one in each of the } k \text{ infinitesimal intervals } (x_i, x_i + dx_i)\}.$$

If μ is discrete then

$$\mathcal{J}_{k,I}(x_1, \ldots, x_k)\, \mu(x_1) \cdots \mu(x_k)$$
$$= \Pr\{\text{there are exactly } k \text{ particles in } I, \text{ one at each of the } k \text{ points } x_i\}.$$

See ref. 12 for details.

For determinantal point processes Janossy densities also have a determinantal form (see ref. 12, p. 140 or ref. 4, Section 2):

$$\mathcal{J}_{k,I}(x_1, \ldots, x_k) = \mathrm{const}(I) \cdot \det(L_I(x_i, x_j))_{i,j=1,\ldots,k}, \tag{6}$$

where

$$L_I = K_I(\mathrm{Id} - K_I)^{-1}. \tag{7}$$

Here the kernel of K_I is the restriction of the kernel $K(x, y)$ to I: $K_I(x, y) = \chi_I(x)\, K(x, y)\, \chi_I(y)$, where $\chi_I(\cdot)$ is the characteristic function of I, and $\mathrm{const}(I)$ is the Fredholm determinant

$$\mathrm{const}(I) = \det(\mathrm{Id} - K_I) = \det(\mathrm{Id} + L_I)^{-1}.$$

The main result of this paper is

Theorem 1. Let $\tilde{\xi}_j$, $j=1,\dots,n$ and $\tilde{\eta}_j$, $j=1,\dots,n$ be biorthonormal bases in $\mathrm{Span}\{\phi_j, j=1,\dots,n\}$ and $\mathrm{Span}\{\psi_j, j=1,\dots,n\}$ considered as subspaces of $L^2(X\setminus I, \mu)$:

$$\tilde{\xi}_k \in \mathrm{Span}(\phi_j, j=1,\dots,n), \quad \tilde{\eta}_k \in \mathrm{Span}(\psi_j, j=1,\dots,n),$$

$$\int_{X\setminus I} \tilde{\xi}_k(x)\, \tilde{\eta}_m(x)\, \mu(dx) = \delta_{km}.$$

Then the kernel of $L_I = K_I(\mathrm{Id}-K_I)^{-1}$ is equal to

$$L_I(x, y) = \sum_{j=1}^{n} \tilde{\xi}_j(x)\, \tilde{\eta}_j(y). \tag{8}$$

The above result readily applies to the so-called $\beta=2$ polynomial ensembles. Such ensembles arise, in particular, in random matrix theory,[10,21] directed percolation and tiling models,[18–20] and representation theory.[4–6] The definition is as follows.

Assume that X is a subset of $\mathbb{R}$ and take

$$p(x_1,\dots,x_n) = \mathrm{const}_n \cdot \prod_{1\leqslant j<k\leqslant n} (x_j - x_k)^2. \tag{9}$$

(Recall that this formula gives the joint distribution density with respect to $\mu(dx_1)\cdots\mu(dx_n)$.)

This is a special case of (1) with $\phi_j(x)=\psi_j(x)=x^{j-1}$, $j=1,\dots,n$. Then we have $\xi_j=\eta_j=p_{j-1}$, where $\{p_j(x)\}$ are normalized orthogonal polynomials on $(X, \mu(dx))$, and $\deg(p_j)=j$. The kernel $K(x, y)$ is the nth Christoffel–Darboux kernel

$$K_n(x, y) = \sum_{j=0}^{n-1} p_j(x)\, p_j(y) = \frac{k_{n-1}}{k_n} \frac{p_n(x)\, p_{n-1}(y) - p_n(y)\, p_{n-1}(x)}{x-y},$$

where k_j is the coefficient of x^j in $p_j(x)$. It should be noted that the kernel K depends on n, but in what follows we will usually omit the subscript n unless this may lead to a confusion.

Clearly, in the case of the polynomial ensemble (9), Theorem 1 states that the kernel of $L_I = K_I(1-K_I)^{-1}$ is the nth Christoffel–Darboux kernel computed for the measure μ restricted to $X\setminus I$. That is,

$$L_I(x, y) = \sum_{j=0}^{n-1} \tilde{p}_j(x)\, \tilde{p}_j(y) = \frac{\tilde{k}_{n-1}}{\tilde{k}_n} \frac{\tilde{p}_n(x)\, \tilde{p}_{n-1}(y) - \tilde{p}_n(y)\, \tilde{p}_{n-1}(x)}{x-y},$$

where

$$\tilde{p}_j(x) = \tilde{k}_j x^j + \{\text{lower order terms}\}, \qquad \int_{X\setminus I} \tilde{p}_k(x)\, \tilde{p}_m(x)\, \mu(dx) = \delta_{km}.$$

One of the particulary nice properties of the Janossy densities is that for any interval (or, more generally, a measurable set) I and non-negative integer k one has

$$\begin{aligned}&\Pr(\text{there are exactly } k \text{ particles in } I)\\ &\quad = \frac{1}{k!} \int_{I^k} \mathcal{J}_{k,I}(x_1,\dots,x_k)\, \mu(dx_1)\cdots\mu(dx_k)\end{aligned}$$

The Janossy densities can be particularly useful in calculating the distribution of the left-most (right-most) particles when the particle space X is a subset of the real line. Indeed, let us denote by $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ the locations of the particles in the increasing order. Then it is easy to see that

$$\begin{aligned}&\Pr(\lambda_k \in (s, s+ds))\\ &\quad = \left(\frac{1}{(k-1)!} \int_{(-\infty,s)^{k-1}} \mathcal{J}_{k,(-\infty,s)}(x_1,\dots,x_{k-1},s)\, \mu(dx_1)\cdots\mu(dx_{k-1})\right)\mu(ds)\\ &\quad = \Pr(\lambda_1 \geqslant s)\, \frac{1}{(k-1)!} \int_{(-\infty,s)^{k-1}} \det(L_{(-\infty,s)}(x_i,x_j))_{i,j=1,\dots,k}\\ &\qquad \times \mu(dx_1)\cdots\mu(dx_{k-1})\,\mu(ds)\end{aligned} \tag{10}$$

$$\begin{aligned}&\Pr(\lambda_1 \geqslant s)\\ &\quad = (\det(\mathrm{Id} + L_{(-\infty,s)}))^{-1} = \det(\mathrm{Id} - K_{(-\infty,s)}),\end{aligned}$$

(where in (10) we put $x_k = s$).

This observation and the Theorem above allow us to compute explicitly the distribution functions of the left-most particles in the hard-edge scaling limit of random matrix models when the parameter (charge at the edge) is equal to zero. We refer to Section 4 below for the details.

The result of Theorem 1 was initially discovered in the case of polynomial ensembles using the techniques of Riemann–Hilbert problems. Later on, it was realized that Theorem 1 has a simpler linear algebraic proof. However, since the main idea of the "Riemann–Hilbert" computation is very useful in deriving Painlevé equations for the distribution of the left- or right-most particles in determinantal point processes, see refs. 2, 3, and 7, we decided to include the argument into this paper; it can be found in Section 2. Section 3 contains the simpler proof. Concluding remarks are given in Section 5.

To conclude the Introduction, let us note that Theorem 1 has a counterpart for the so-called pfaffian ensembles. (The $\beta=1$ and 4 (or "orthogonal" and "symplectic") random matrix ensembles are the most known examples of the pfaffian ensembles.) See the companion paper[28] for details.

2. RIEMANN–HILBERT PROBLEM

In this section we will briefly describe two applications of the Riemann–Hilbert problem (to computing orthogonal polynomials and to inverting integrable integral operators) and use them to derive Theorem 1 in the case of polynomials ensembles. Since we use the Riemann–Hilbert problem (RHP, for short) mainly for instructional purposes, we avoid the discussion of any technical issues involved.

Let Σ be an oriented contour in $\mathbb{C}$. We agree that when we go along the contour in the direction of orientation, the positive side lies to the left and the negative side lies to the right. Let v be a map from Σ to $\mathbf{GL}(l, \mathbb{C})$, where $l=1, 2,\ldots$. We say that an $l\times l$ matrix function $m=m(z)$ is a solution of the RHP (Σ, v) if[9, 10]

$$\text{(i)} \quad m(z) \text{ is analytic in } \mathbb{C}\setminus\Sigma, \tag{11}$$

$$\text{(ii)} \quad m_+(z)=m_-(z)\,v(z),\ z\in\Sigma. \tag{12}$$

Here $m_+(z), m_-(z)$ stand for the limiting values of $m(z)$ as z approaches Σ from the positive (negative) side. If, in addition, $m(z)\to \mathrm{Id}$ as $z\to\infty$ then we say that $m(z)$ solves the normalized RHP (Σ, v). The matrix v is usually called the jump matrix for the RHP.

First we describe the connection of RHP to orthogonal polynomials, see refs. 14 and 15. Let $d\mu(x)=\omega(x)\,dx$ be an absolutely continuous measure on the real line, such that the non-negative density ω decays at infinity sufficiently fast (in particular, all moments exist). Consider an RHP on $\mathbb{R}$ oriented from left to right with the jump matrix

$$v(z)=\begin{pmatrix} 1 & \omega(z) \\ 0 & 1 \end{pmatrix}. \tag{13}$$

Fix a non-negative integer n. We are looking for a solution of the RHP $(\mathbb{R}, v)$ satisfying

$$m(z)=(\mathrm{Id}+O(z^{-1}))\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \qquad z\to\infty.$$

It appears that this RHP has a unique solution given by

$$m(z)=\begin{pmatrix} \pi_n(z) & (C(\omega\pi_n))(z) \\ \gamma_{n-1}\pi_{n-1}(z) & \gamma_{n-1}(C(\omega\pi_{n-1}))(z) \end{pmatrix}, \qquad z\notin\mathbb{R}, \tag{14}$$

where $\pi_n(z)=z^n+\cdots$ is the nth monic orthogonal polynomial corresponding to the weight function $\omega(x)$,

$$(Ch)(z)=\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{h(\xi)}{\xi-z}\,d\xi$$

is the Cauchy transform, $\gamma_n=-2\pi i k_n^2$, and k_n is the leading coefficient of the nth orthonormal polynomial p_n, i.e., $p_n(z)=k_n\pi_n(z)$. Thus, computing the orthogonal polynomials with the weight $\omega(z)$ is equivalent to solving RHP of the form above.

Now let us explain the relation of RHP to integrable operators. Let I be a subset of $\mathbb{R}$ (typically, a disjoint union of finitely many intervals). We recall that an integral operator M in $L^2(I, dx)$ with the kernel $M(x, y)$ is called integrable[11, 23, 24] if

$$M(x, y)=\frac{\sum_{i=1}^{l} f_i(x)\, g_i(y)}{x-y} \tag{15}$$

for some $l=2, 3,...$ and some functions f_i, g_i on I. We assume that $\sum_{i=1}^{l} f_i(x)\, g_i(x)=0$ so that the kernel has no singularity on the diagonal.

In particular, the formula for the Christoffel–Darboux kernel in the case of polynomial ensembles discussed in Section 1 means that the operators K and K_I can be viewed as integrable operators in $L^2(\mathbb{R}, dx)$ and $L^2(I, dx)$ with $l=2$, and we may take

$$f_1(x)=\frac{1}{2\pi i k_n}\,p_n(x)\sqrt{\omega(x)}, \qquad f_2(x)=-k_{n-1}\,p_{n-1}(x)\sqrt{\omega(x)}, \tag{16}$$

$$g_1(x)=2\pi i k_{n-1}\,p_{n-1}(x)\sqrt{\omega(x)}, \qquad g_2(x)=\frac{1}{k_n}\,p_n(x)\sqrt{\omega(x)}. \tag{17}$$

The appearance of $\sqrt{\omega(x)}$ has to do with the fact that we consider the Lebesgue measure rather than $\mu(dx)$ as our reference measure for the L^2-space.

It turns out that if the operator $\mathrm{Id}-M$ is invertible then the resolvent $R=M(\mathrm{Id}-M)^{-1}$ is also an integrable operator and

$$R(x, y) = \frac{\sum_{i=1}^{l} F_i(x)\, G_i(y)}{x - y}, \tag{18}$$

$$F_i = (\mathrm{Id} - M)^{-1} f_i, \quad G_i = (\mathrm{Id} - M^t)^{-1} g_i, \qquad i = 1, 2, \dots, l, \tag{19}$$

see refs. 11, 23, and 24 for a very nice exposition. Furthermore, the functions F_i and G_i can be obtained through solving a RHP as follows. Let v' be an $l \times l$ matrix valued function on I given by

$$v' = \mathrm{Id} - 2\pi i f g^t, \qquad f = (f_1, \dots, f_l)^t, \quad g = (g_1, \dots, g_l)^t. \tag{20}$$

One can prove[11, 23, 24] that the normalized RHP (I, v') has a unique solution $m'(z)$, and

$$F = (F_1, \dots, F_l)^t = (m')_{\pm} f, \tag{21}$$

$$G = (G_1, \dots, G_l)^t = (m')_{\pm}^{-t} g. \tag{22}$$

The following observation is crucial.

Lemma 1. Let m be the solution (14) of the RHP $(\mathbb{R}, v)$ and let m' be the solution of the normalized RHP (I, v') with v' given by (16), (17), and (20). Then $M = m'm$ solves the RHP $(\mathbb{R} \setminus I, v)$ with the asymptotics $\mathrm{diag}(z^n, z^{-n})$ as $z \to \infty$, and hence

$$M(z) = \begin{pmatrix} \tilde{\pi}_n(z) & (C(\omega \chi_{\mathbb{R}\setminus I} \tilde{\pi}_n))(z) \\ \tilde{\gamma}_{n-1} \tilde{\pi}_{n-1}(z) & \tilde{\gamma}_{n-1} (C(\omega \chi_{\mathbb{R}\setminus I} \tilde{\pi}_{n-1}))(z) \end{pmatrix}, \qquad z \notin \mathbb{R} \setminus I, \tag{23}$$

where $\sim$ signifies that the corresponding polynomials are orthogonal on $\mathbb{R} \setminus I$ with respect to the same weight function ω.

The proof of this lemma is based on Lemma 4.3 of ref. 7 (see also Lemma 2.4 of ref. 2 for a discrete analog). The analog of Lemma 1 for weight functions with discrete support was one of the basic tools used in ref. 3.

Proof. A straightforward calculation shows that on I we have $v' = m_+ v^{-1} m_+^{-1} = m_- v^{-1} m_-^{-1}$. Thus on I

$$M_-^{-1} M_+ = m_-^{-1} (m'_-)^{-1}\, m'_+ m_+ = m_-^{-1} v' m_+ = v^{-1} m_-^{-1} m_+ = \mathrm{Id}.$$

On the other hand, since $m'(z)$ is holomorphic away from I and tends to Id as $z \to \infty$, it is clear that on $\mathbb{R} \setminus I$, $M(z)$ satisfies the same jump condition as $m(z)$, and that it also has the same asymptotics as $m(z)$ when $z \to \infty$. ∎

Proof of Theorem 1. We apply the formalism described above to the Christoffel–Darboux kernel with f_i, g_i specialized by (16) and (17) above. As before, we will use the notation $\tilde{p}_k$, $\tilde{\pi}_k$ for the kth orthonormal and monic orthogonal polynomials corresponding to the weight ω on $\mathbb{R}\backslash I$, and we also denote

$$q_k = C(\omega\pi_k), \qquad \tilde{q}_k = C(\omega\chi_{\mathbb{R}\backslash I}\tilde{\pi}_k).$$

In the calculations below we use the identity $\det m(z) \equiv \det M(z) \equiv 1$. Indeed, Liouville's theorem readily implies that if the jump matrix of a RHP has determinant 1 and the determinant of the asymptotics of a solution at infinity is also equal to 1, then the determinant of any solution of this RHP (having the corresponding asymptotics at infinity) must equal 1 identically.

We have

$$\begin{aligned}
F_1 &= (m'f)_1 = (Mm^{-1})_{11} f_1 + (Mm^{-1})_{12} f_2 \\
&= M_{11}m_{22} f_1 - M_{12}m_{21} f_1 - M_{11}m_{12} f_2 + M_{12}m_{11} f_2 \\
&= \frac{k_{n-1}}{k_n\tilde{k}_n}\left(-k_{n-1}\tilde{p}_n q_{n-1} p_n + \tilde{k}_n\tilde{q}_n p_{n-1} p_n + k_{n-1}\tilde{p}_n q_n p_{n-1} - \tilde{k}_n\tilde{q}_n p_n p_{n-1}\right)\sqrt{\omega} \\
&= \frac{k_{n-1}^2}{k_n\tilde{k}_n}\left(q_n p_{n-1} - q_{n-1} p_n\right)\tilde{p}_n\sqrt{\omega} = \frac{1}{2\pi i\tilde{k}_n}\det(m)\,\tilde{p}_n\sqrt{\omega} = \frac{1}{2\pi i\tilde{k}_n}\tilde{p}_n\sqrt{\omega}.
\end{aligned}$$

Similar calculations yield

$$G_1 = 2\pi i\tilde{k}_{n-1}\tilde{p}_{n-1}\sqrt{\omega}, \qquad F_2 = -\tilde{k}_{n-1}\tilde{p}_{n-1}\sqrt{\omega}, \qquad G_2 = \frac{1}{\tilde{k}_n}\tilde{p}_n\sqrt{\omega}.$$

Hence, the kernel of $L_I = K_I(\mathrm{Id} - K_I)^{-1}$ equals

$$\begin{aligned}
L_I(x,y) &= \frac{F_1(x)\,G_1(y) + F_2(x)\,G_2(y)}{x-y} \\
&= \frac{\tilde{k}_{n-1}}{\tilde{k}_n}\frac{\tilde{p}_n(x)\tilde{p}_{n-1}(y) - \tilde{p}_n(y)\tilde{p}_{n-1}(x)}{x-y}\sqrt{\omega(x)\,\omega(y)}.
\end{aligned}$$

Recall that the factor $\sqrt{\omega(x)\,\omega(y)}$ is due to the fact that we are working in $L^2(I,dx)$ rather than $L^2(I,\omega(x)\,dx)$. The proof of Theorem 1 for polynomial ensembles is complete. ▮

3. LINEAR ALGEBRAIC PROOF

We use the notation of Section 1. Consider the integral operators K_I and L_I in $L^2(I, \mu)$ with the kernels

$$K_n(x, y) = \sum_{j=1}^{n} \xi_j(x)\, \eta_j(y) \tag{24}$$

and

$$L_I(x, y) = \sum_{j=1}^{n} \tilde{\xi}_j(x)\, \tilde{\eta}_j(y). \tag{25}$$

Both operators are finite-dimensional:

$$\operatorname{Ran}(K_I) = \operatorname{Ran}(L_I) = \mathscr{H}_1 := \operatorname{Span}(\phi_j) = \operatorname{Span}(\xi_j) = \operatorname{Span}(\tilde{\xi}_j),$$
$$\operatorname{Ker}(K_I) = \operatorname{Ker}(L_I) = \mathscr{H}_2^{\perp}, \qquad \mathscr{H}_2 := \operatorname{Span}(\psi_j) = \operatorname{Span}(\eta_j) = \operatorname{Span}(\tilde{\eta}_j),$$

where the index j ranges over $\{1,\dots, n\}$, and the subspaces are taken inside $L^2(I, \mu)$. Therefore, in order to prove that $L_I = K_I(\mathrm{Id} - K_I)^{-1}$, it is enough to prove this relation for the restrictions of K_I and L_I to the n-dimensional space $\mathscr{H}_1$. To this end we compute the matrices of the restrictions of the operators K_I, L_I to $\mathscr{H}_1$ in the basis $\{\xi_j\}_{j=1,\dots,n}$.

Let us denote by G_I and $G_{X\setminus I}$ the $n \times n$ matrices with entries

$$\begin{aligned}(G_I)_{jk} &= \int_I \xi_j(x)\, \eta_k(x)\, \mu(dx), \\ & \qquad\qquad\qquad\qquad j, k = 1,\dots, n. \\ (G_{X\setminus I})_{jk} &= \int_{X\setminus I} \xi_j(x)\, \eta_k(x)\, \mu(dx)\end{aligned}$$

Since $\{\xi_j\}$, $\{\eta_k\}$ are biorthonormal on X, we have $G_I + G_{X\setminus I} = \mathrm{Id}$. The matrix of K_I on $\mathscr{H}_1$ in the basis $\{\xi_j\}$ is given by G_I. To calculate the matrix of the restriction of L_I on $\mathscr{H}_1$ we biorthonormalize the functions ξ_j, η_j, $j = 1,\dots, n$, in $L^2(X \setminus I, \mu)$. This gives (cf. Proposition 2.2 in ref. 1)

$$\begin{aligned}\sum_{j=1}^{n} \tilde{\xi}_j(x)\, \tilde{\eta}_j(y) &= \sum_{j,k=1}^{n} \xi_j(x)\, \eta_k(y) (G_{X\setminus I})^{-1}_{kj} \\ &= \sum_{j,k=1}^{n} \xi_j(x)\, \eta_k(y) (\mathrm{Id} - G_I)^{-1}_{kj}.\end{aligned} \tag{26}$$

It immediately follows from (25) that the matrix of the restriction of L_I to $\mathscr{H}_1$ in the basis $\{\xi_j\}$ is equal to $G_I(\mathrm{Id} - G_I)^{-1}$. The proof is complete.

4. HARD EDGE WITH ZERO CHARGE

In a few special cases, the polynomials orthogonal with respect to ω on $X\setminus I$ can be easily expressed in terms of the orthogonal polynomials on X. We consider the Laguerre ensemble of positive definite matrices as an example.

Every positive definite $n\times n$ matrix M can be written (in a non-unique way) as $M=AA^*$, where A is an $n\times n$ matrix with complex entries and A^* is the adjoint matrix. The probability measure in the Laguerre ensemble (also called Wishart ensemble in statistics) is defined as:[8]

$$P(dM)=Z_n^{-1}\exp(-\mathrm{Tr}(AA^*))\det(AA^*)^{\alpha}\,dA, \tag{27}$$

where dA is the Lebesgue measure on the $2n^2$-dimensional space of $n\times n$ complex matrices, Z_n^{-1} is a normalization constant, and $\alpha>-1$. The joint probability density of the distribution of the eigenvalues of $M=AA^*$ is equal to

$$\begin{gathered}p(x_1,\ldots,x_n)=\mathrm{const}_n\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)^2\prod_{j=1}^{n}x_j^{\alpha}e^{-x_j},\\ x_j\in(0,+\infty),\quad j=1,\ldots,n.\end{gathered} \tag{28}$$

The polynomials orthogonal with the weight $\omega(x)=x^{\alpha}e^{-x}$ on $\mathbb{R}_+=(0,+\infty)$ are the classical Laguerre polynomials (see, e.g., ref. 13). In the special case of $\alpha=0$ and $I=(0,t)$, the orthogonal polynomials on $X\setminus I=\mathbb{R}_+\setminus(0,t)=[t,+\infty)$ are obtained from the Laguerre polynomials by the simple shift of variable $x\mapsto x-t$, i.e., $\tilde p_j(x)=p_j(x-t)$, $j=0,1,\ldots$. Thus, by Theorem 1, the kernel of $L_I^{\mathrm{Lag}(n)}=K_I^{\mathrm{Lag}(n)}(\mathrm{Id}-K_I^{\mathrm{Lag}(n)})^{-1}$ is equal to

$$L_I^{\mathrm{Lag}(n)}(x,y)=K^{\mathrm{Lag}(n)}(x-t,y-t), \tag{29}$$

where $K^{\mathrm{Lag}(n)}(x,y)$ is the order n Christoffel–Darboux kernel for Laguerre polynomials with $\alpha=0$.

It is well known, see refs. 16, 25, and 29, that when n becomes large, the smallest eigenvalues in the Laguerre ensemble are of order n^{-1}. Moreover, if we rescale all the eigenvalues of the nth Laguerre ensemble by n^{-1} then there exists a scaling limit as $n\to\infty$ of all the correlation functions. (In the random matrix theory this procedure is usually referred to as "hard edge scaling limit.") The limit correlation functions also have the determinantal form (5) with the so-called Bessel kernel:

$$\lim_{n\to\infty} n^{-k}\rho_k^{\mathrm{Lag}(\alpha,n)}\left(\frac{x_1}{n},\dots,\frac{x_k}{n}\right)=\det(K^{(\alpha)}(x_i,x_j))_{i,j=1,\dots,k},$$

$$K^{(\alpha)}(x,y)=\frac{J_\alpha(2\sqrt{x})\sqrt{y}\,J'_\alpha(2\sqrt{y})-J_\alpha(2\sqrt{y})\sqrt{x}\,J'_\alpha(2\sqrt{x})}{x-y}$$

$$=\int_0^1 J_{\alpha+1}(2\sqrt{\tau x})\,J_{\alpha+1}(2\sqrt{\tau y})\,d\tau$$

$$=\sum_{k,l=0}^{\infty}\frac{(-1)^k x^{k+\alpha/2}}{k!\,\Gamma(\alpha+k+1)}\frac{(-1)^l y^{l+\alpha/2}}{l!\,\Gamma(\alpha+l+1)}\frac{1}{\alpha+k+l+1}.$$

Here $J_\nu(\cdot)$ is the J-Bessel function, see, e.g., ref. 13. Note that if $\alpha=0$ then the above formula makes sense for any $x, y\in\mathbb{C}$.

Proposition 1. For any $s>0$, let $K_s^{(0)}$ be the (bounded) integral operator in $L^2((0,s),dx)$ defined by the restriction of the Bessel kernel $K^{(\alpha)}(x,y)$ with $\alpha=0$ to $(0,s)\times(0,s)$. Then the operator $K_s^{(0)}(1-K_s^{(0)})^{-1}$ is bounded and has a kernel which is equal to $K^{(0)}(x-s,y-s)$.

Proof. The relation (29) implies

$$K^{\mathrm{Lag}(n)}(x,y)=K^{\mathrm{Lag}(n)}(x-t,y-t)-\int_0^t K^{\mathrm{Lag}(n)}(x-t,u-t)\,K^{\mathrm{Lag}(n)}(u,y)\,du,$$

$$x,y\in(0,t). \tag{30}$$

Since $n^{-1}K^{\mathrm{Lag}(n)}(xn^{-1},yn^{-1})$ tends to $K^{(0)}(x,y)$ as $n\to\infty$ uniformly on compact subsets of $\mathbb{C}$ (this follows, e.g., from the proof of Theorem 4.5 in ref. 1), taking the scaling limit in (30) yields

$$K^{(0)}(x,y)=K^{(0)}(x-s,y-s)-\int_0^s K^{(0)}(x-s,u-s)\,K^{(0)}(u,y)\,du,$$

$$x,y\in(0,s). \tag{31}$$

Denote by $L_s^{(0)}$ the operator in $L^2((0,s),dx)$ with the kernel $L_s^{(0)}(x,y)=K^{(0)}(x-s,y-s)$. Since the kernel of this operator is the uniform limit of the kernels of the nonnegative operators $L_I^{\mathrm{Lag}(n)}=K^{\mathrm{Lag}(n)}(x-t,y-t)$, we have $L_s^{(0)}\geqslant 0$, and hence -1 does not belong to the spectrum of $L_s^{(0)}$. Thus, (31) can be rewritten in the form $K_s^{(0)}=L_s^{(0)}(\mathrm{Id}+L_s^{(0)})^{-1}$, and this is equivalent to the statement of the proposition. ∎

Corollary 1. Let $\lambda_1^{(n)} \leqslant \lambda_2^{(n)} \leqslant \cdots \leqslant \lambda_n^{(n)}$ be the ordered eigenvalues of the Laguerre ensemble (28) with $\alpha = 0$. Then

$$\Pr\left(\lambda_1^{(n)} \geqslant \frac{s}{n}\right) = e^{-s}, \tag{32}$$

$$\lim_{n\to\infty} \Pr\left(\lambda_{k+1}^{(n)} \geqslant \frac{s}{n}\right) = e^{-s} \int_{(-s,0)^k} \det(K^{(0)}(x_i, x_j))_{i,j=1,\ldots,k}\, dx_1 \cdots dx_k, \qquad k \geqslant 2. \tag{33}$$

In particular,

$$\lim_{n\to\infty} \Pr\left(\lambda_2^{(n)} \geqslant \frac{s}{n}\right) = \frac{e^{-s}}{2} \int_0^{2\sqrt{s}} x(I_0^2(x) - I_1^2(x))\, dx, \tag{34}$$

where $I_\nu(\cdot)$ is the I-Bessel function.

The formula (32) was first observed in ref. 16. The limiting distribution (34) of the second smallest eigenvalue was computed in refs. 17 and 29.[3] Further results in this direction, including formulas similar to (33) can be found in ref. 32.

Proof. The relation (32) is easy:

$$\Pr\left(\lambda_1^{(n)} \geqslant \frac{s}{n}\right) = \frac{\int_{(s/n,+\infty)^n} \prod_{i<j} (x_i - x_j)^2 \prod_j e^{-x_j}\, dx_j}{\int_{(0,+\infty)^n} \prod_{i<j} (x_i - x_j)^2 \prod_j e^{-x_j}\, dx_j}$$
$$= \frac{\int_{(0,+\infty)^n} \prod_{i<j} (x_i - x_j)^2 \prod_j e^{-x_j - s/n}\, dx_j}{\int_{(0,+\infty)^n} \prod_{i<j} (x_i - x_j)^2 \prod_j e^{-x_j}\, dx_j} = e^{-s}.$$

The relation (33) follows from (10) applied to the Laguerre ensemble and the uniform convergence of kernels mentioned in the proof of the proposition above. Finally, using the L'Hôpital rule we obtain

$$\int_0^s K^{(0)}(-x, -x)\, dx$$
$$= \int_0^s \left((J_0'(2i\sqrt{x}))^2 - \frac{J_0(2i\sqrt{x})\, J_0'(2i\sqrt{x})}{2i\sqrt{x}} - J_0(2i\sqrt{x})\, J_0''(2i\sqrt{x}) \right) dx.$$

The formulas

$$J_0'(z) = -J_1(z), \qquad J_0''(z) = z^{-1} J_1(z) - J_0(z),$$
$$I_0(z) = J_0(iz), \qquad I_1(z) = -iJ_1(iz),$$

[3] Note that the integral (34) can be evaluated in terms of Bessel functions, see, e.g., (2.30) in ref. 29.

see ref. 13, and the change of variable $x \mapsto x^2/2$ bring the last integral to the form (34). ∎

Of course, if $\lambda_1 < \lambda_2 < \cdots$ are the ordered particles of the determinantal point process with the correlation functions given by the Bessel kernel with $\alpha = 0$ then the right–hand sides of (32) and (33) are equal to $\Pr(\lambda_1 \geqslant s)$ and $\Pr(\lambda_{k+1} \geqslant s)$, respectively.

The calculations similar to those above can be done for the Jacobi ensemble corresponding to $\omega(x) = (1-x)^\alpha (1+x)^\beta$, $x \in (-1, 1)$, in the special cases $I = (t, 1)$, $\alpha = 0$; $I = (-1, t)$, $\beta = 0$. After appropriate rescaling one again obtains the limit relations of the form (32), (33), (34).

5. CONCLUDING REMARKS

Take two n-point ensembles with joint probability densities of the form $\mathrm{const} \cdot p_n(x_1,..., x_n)$ with the same p_n, but assume that these two ensembles are supported by different sets—the first one lives on (X, μ) while the second one lives on $(X \setminus I, \mu)$, where I is a subset of X. Of course, the normalization constants for these two ensembles will be different. The kth Janossy density $\mathcal{J}_{k,I}(x_1,..., x_k)$ of the first ensemble is given by the formula

$$\mathcal{J}_{k,I}(x_1,..., x_k) = \mathrm{const}' \cdot \int_{(X\setminus I)^{n-k}} p_n(x_1,..., x_n)\, \mu(dx_{k+1}) \cdots \mu(dx_n),$$

$$x_1,..., x_k \in I,$$

while the kth correlation function $\tilde{\rho}_k(x_1,..., x_k)$ of the second ensemble equals

$$\tilde{\rho}_k(x_1,..., x_k) = \mathrm{const}'' \cdot \int_{(X\setminus I)^{n-k}} p_n(x_1,..., x_n)\, \mu(dx_{k+1}) \cdots \mu(dx_n),$$

$$x_1,..., x_k \in X \setminus I.$$

The only difference between the two formulas above is in the constant prefactor and in the domain where $x_1,..., x_k$ are allowed to vary. Clearly, this suggests that there should be a direct relation between $\mathcal{J}_{k,I}$ and ρ_k, and in the case of determinantal ensembles such relation is provided by Theorem 1.[4]

[4] Note, however, that a simple comparison of the two formulas above does not give a proof of Theorem 1 because these formulas only provide the symmetric minors of the corresponding kernels.

Since the argument above does not depend on the specific form of the density p_n, one might expect that Theorem 1 should have an analog for the pfaffian ensembles (see, e.g., refs. 26, 30, and 31 for definitions). This is exactly the case, and the corresponding result is presented in the companion paper.[(28)]

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